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Decay and scattering of solutions for a generalized Boussinesq equation

Ying Wang [a], Chunlai Mu [b,*], Yonghong Wu [c]

[a] *School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, 610054, PR China*
[b] *College of Mathematics and Physics, Chongqing University, Chongqing, 400044, PR China*
[c] *Department of Mathematics and Statistics, Curtin University of Technology, Perth, 6845 WA, Australia*

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ABSTRACT

In this paper, we consider the long-time behavior of small solutions of the Cauchy problem for a generalized Boussinesq equation. A scattering operator and the nonlinear scattering for small amplitude solutions of the Boussinesq equation are established under certain hypotheses.

1. Introduction

The subject of this paper is to study the long-time behavior of solutions and nonlinear scattering theory for a generalized Boussinesq equation

$$u_{tt} - u_{xxtt} + u_{xxxxtt} = -u_{xxxx} + u_{xx} + f(u)_{xx}. \tag{1.1}$$

This model arises in some physical problems. In [15], the authors studied a class of Boussinesq equations, as below, which models the water wave problem with surface tension

$$u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{xxxxtt} + \left(u^2\right)_{xx}, \tag{1.2}$$

* Corresponding author.
E-mail address: clmu2005@163.com (C.L. Mu).

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where $x, t, \mu \in R$ and $u(t,x) \in R$. The model can also be formally derived from the 2D water wave problem. For a degenerate case, it has been proved that the long wave limit can be described approximately by two decoupled Kawahara-equations. A more natural model is the extension of the classical Boussinesq equation as follows (see also [2])

$$u_{tt} = u_{xx} + (\mu + 1)u_{xxxx} - u_{xxxxxx} + \left(u^2\right)_{xx}. \tag{1.3}$$

In [15], the authors included the term with the sixth order derivatives since they were interested precisely in the case where the coefficient of the term with the fourth order derivative is small, i.e. $1 + \mu = \varepsilon^2 \nu$, with $\nu \in R$ fixed. The lowest order nonlinear terms in the water wave problem remain unchanged from the classical equation because they are independent of surface tension. The model (1.2) is ill-posed, and thus they prefer (1.3) which is an equivalent model in the sense that in the derivation of this model there is a formal equivalence of second order time and second order space derivatives in the long-wavelength limit, i.e. $\partial_t^2 u = \partial_x^2 u + O(\varepsilon^2)$.

The classical Boussinesq equations and its generalization have been studied from various points of view [1,3–9,11–14,16–23]. For the following Boussinesq equation

$$u_{tt} = u_{xxxx} + u_{xx} - f(u)_{xx}, \tag{1.4}$$

Liu [9–12] studied the local existence and blow-up, instability and strong instability of solitary-waves, long-time behavior of solutions and nonlinear scattering theory of (1.4). In [17–20], Varlamov studied the following damped Boussinesq equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta\left(u^2\right)_{xx}, \tag{1.5}$$

where the second term on the left-hand side is responsible for dissipation. Under various assumptions on initial boundary data, the author has constructed the classical solution of the problem and obtained the long-time asymptotics in explicit form. Using the eigenfunction expansion method, the author also studied the long-time asymptotics of a damped Boussinesq equation which is a multi-dimensional generalization of (1.5).

In [22,23], Wang and Chen studied the multidimensional Boussinesq equation

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u = \Delta f(u), \quad (x,t) \in R^n \times (0, +\infty), \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in R^n. \end{cases} \tag{1.6}$$

The existence and uniqueness of the global solution and the blow-up of the solution for (1.6) have been proved. The global existence of the strong solution and small amplitude solution for (1.6) have also been obtained.

In [24], the authors obtain the existence and uniqueness of the local solutions of (1.1). For a class of nonlinear perturbations, blow-up solutions are obtained. Furthermore, the global existence and nonlinear scattering for small amplitude solutions are established. For the problem (1.1) with several variables, the existence and the uniqueness of the global solution for the Cauchy problem of (1.1) are obtained in R^n. And the blow-up of the solution for (1.1) is proved by Wang [25] in R^n. Moreover, in [26], the global existence and nonlinear scattering for small amplitude solutions are obtained in R^n.

In this paper, we consider the long-time behavior of small solutions of the problem (1.1). Our purpose is to study the scattering of the solution of (1.1). At first, we write (1.1) as an integral equation and treat the nonlinearity as a small perturbation of the linear part of the equation. We use an estimate for the uniform decay of solutions of the linearized equation of (1.3) to obtain a priori estimate on time-weighted norms of solution $\vec{u}$ of (1.1). Then, we construct a scattering operator S. The scattering operator is defined as $S(\vec{h}_-) = \vec{h}_+$, where $S(t)\vec{h}_\pm = \vec{u}_\pm(t)$ satisfies

$$\left\|\vec{u}(t) - \vec{u}_\pm(t)\right\|_{Y_1} \to 0, \tag{1.7}$$

as $t \to \pm\infty$, with the initial data $\vec{h}_\pm$ and Y_1 is the energy space defined as below. The scattering theory for the Boussinesq equation was studied by many authors. In [24], the nonlinear scattering for small amplitude solutions of (1.1) is established. But the authors only obtained the scattering result as $t \to +\infty$. In this paper, we will obtain the scattering results as $t \to \pm\infty$ by establishing estimates different from that of [24].

Firstly, we rewrite (1.1) with the initial data $\vec{u}_0$ as a system of equations, namely,

$$\frac{d\vec{u}}{dt} = A\vec{u} + N(\vec{u}), \tag{1.8}$$

where

$$\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \partial_x \\ \frac{(I-\partial_x^2)\partial_x}{I-\partial_x^2+\partial_x^4} & 0 \end{pmatrix}, \qquad N(\vec{u}) = \begin{pmatrix} 0 \\ \frac{\partial_x}{I-\partial_x^2+\partial_x^4}(f(u)) \end{pmatrix}, \qquad \vec{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Throughout this paper, we use $|\cdot|_p$ to denote the norm in $L^p(R)$ and $\|\cdot\|_{s,p}$ for the norm in the Sobolev space $W^{s,p}(R)$, $\|\cdot\|_s$ for the norm in the Sobolev space $H^s(R)$, $\|\cdot\|_0 = |\cdot|_2$. Let $\|\vec{u}\|_{s,p} = \|(u,v)\|_{s,p} = \|u\|_{s,p} + \|v\|_{s+1,p}$ for real s, $1 \leqslant p \leqslant +\infty$ denote the norm of the space $Y_{s,p} = L_s^p \times L_{s+1}^p$ where $L_s^p = \Lambda^{-s}L^p$ denotes the Bessel potential space with potential $\Lambda^s = (I - \partial_x^2)^{\frac{s}{2}}$ whose norm will be denoted by $\|\cdot\|_{s,p} = |\Lambda^s \cdot|_p$. When $p = 2$, we will write H^s instead of L_s^2, with the norm $\|\cdot\|_s$, and $Y_s = H^s \times H^{s+1}$ with the norm $\|\vec{u}\|_s = \|u\|_s + \|v\|_{s+1}$.

At first, by using the contraction mapping theorem, we obtain the following existence of the local solution to (1.8).

Theorem 1.1. *(See [24].) Let $\vec{u}_0 = (u_0, v_0) \in Y_1$ and $f \in C^m$, $f(0) = 0$ for some integer $m \geqslant s$. Then there exists $T > 0$ and a unique solution $\vec{u} = (u, v)$ of (1.8) in $C([0,T); Y_1)$ with $\vec{u}(0) = \vec{u}_0$. The time T is independent of s. Moreover, the interval of existence can be extended to a maximal interval $[0, T_{\max})$ such that either*

(i) $T_{\max} = +\infty$, *or*
(ii) $T_{\max} < +\infty$ *and* $\lim_{t\to T_{\max}^-} \|\vec{u}(t)\|_{Y_1} = +\infty$.

Furthermore, the energy $E(\vec{u})$ and the momentum $M(\vec{u})$ of (1.8) are conserved for $0 \leqslant t < T_{\max}$, where

$$E(\vec{u}) = \int_R \left(\frac{1}{2}v^2 + \frac{1}{2}v_x^2 + \frac{1}{2}v_{xx}^2 + \frac{1}{2}u^2 + \frac{1}{2}u_x^2 + F(u)\right) dx,$$

$$M(\vec{u}) = \int_R (uv + u_x v_x + u_{xx} v_{xx})\, dx$$

and $F(u) = \int_0^u f(\tau)\, d\tau$. In addition, if $\vec{u}_0 \in Y_s$ and $m \geqslant s \geqslant 1$, then $\vec{u} \in C^1([0, T_{\max}), Y_s)$.

The next theorem is the main result in this paper.

Theorem 1.2. *Assume that $\Lambda^2 u_0 \in L^1$ and $\Lambda^1 v_0 \in L^1$, where $\Lambda^2 = (I - \partial_x^2)$. Let $f : R \to R$ is a C^1 function with $|f(s)| = O(|s|^p)$ and $f'(s) = O(|s|^{p-1})$ as $s \to 0$, for $p > 1$. Also, assume $\vec{u} = (u, v)$ satisfies (1.8) with $\vec{u}(0) = \vec{u}_0$. Then*

$$\sup_{-\infty<x<+\infty} \left| \int_{-\infty}^{x} \vec{u}(z,t)\, dz \right| \leqslant C_0 t^{1-\nu(p-1)}$$

for $0 < t < T$, *where* T *denotes the maximum existence time for* $\vec{u}$, *the constant* C_0 *depends only on* $\|\vec{u}_0\|_{Y_1}$, f *and* $\sup_{0\leqslant t<T}\|\vec{u}(\cdot,t)\|_{Y_1}$,

$$\nu = \begin{cases} \frac{1}{2s+6} & \text{for } s \geqslant \frac{3}{2}, \\ \frac{2s-1}{2(2s+6)} & \text{for } s \leqslant \frac{3}{2}. \end{cases}$$

Next, we shall construct the scattering operator S. We rewrite (1.8) in its integral form

$$\vec{u}(t) = S(t)\vec{h}_- + \int_s^t S(t-\tau)\vec{f}\big(\vec{u}(\tau)\big)\,d\tau, \tag{1.9}$$

$$\vec{u}(t) = S(t)\vec{h}_+ + \int_s^t S(t-\tau)\vec{f}\big(\vec{u}(\tau)\big)\,d\tau, \tag{1.10}$$

where $\vec{u}$ is the solution of (1.8) with different initial value $\vec{u}_\pm(s) = S(s)\vec{h}_\pm$ at $t = s$. Letting $s \to \pm\infty$, we have the following integral equations which relate $\vec{u}$ to $\vec{h}_\pm$

$$\vec{u}(t) = S(t)\vec{h}_- + \int_{-\infty}^t S(t-\tau)\vec{f}\big(\vec{u}(\tau)\big)\,d\tau, \tag{1.11}$$

$$\vec{u}(t) = S(t)\vec{h}_+ + \int_{+\infty}^t S(t-\tau)\vec{f}\big(\vec{u}(\tau)\big)\,d\tau. \tag{1.12}$$

We then introduce the norm

$$\|\vec{u}\|_V = \sup_{t\in R}\big\{\big(1+|t|\big)^{\nu(1-\frac{2}{p+1})}\big\|\vec{u}(t)\big\|_{1,p+1} + \big\|\vec{u}(t)\big\|_1\big\} \tag{1.13}$$

and the space $V = \{\vec{u} \in C(R, Y_{1,p+1} \cap Y_1) \mid \|\vec{u}\|_V < \infty\}$.

Theorem 1.3. *Let* $f \in C^1(R)$ *satisfy* $|f(\tau)| = O(|\tau|^p)$ *and* $|f'(\tau)| = O(|\tau|^{p-1})$ *as* $\tau \to 0$, *and let* $\nu(1 - \frac{2}{p+1})p > 1$. *Assume* $\vec{h}_- \in Y_1$ *and* $S(\cdot)\vec{h}_- \in V$. *Then there exists* $\delta > 0$ *such that if* $\|S(\cdot)\vec{h}_-\|_V < \delta$, *there exists a unique solution* $\vec{u}$ *of* (1.11) *in* $C(R; Y_{1,p+1} \cap Y_1)$ *and*

$$\big\|\vec{u}(t) - S(t)\vec{h}_-\big\|_1 \to 0 \tag{1.14}$$

as $t \to -\infty$. *Furthermore, there exists a unique* $\vec{u} \in Y_1$ *such that*

$$\big\|\vec{u}(t) - S(t)\vec{h}_+\big\|_1 \to 0 \tag{1.15}$$

as $t \to +\infty$. *In addition,*

$$\big\|u(t)\big\|_1^2 + \big\|v(t)\big\|_2^2 - 2\int_{-\infty}^{\infty} F\big(u(t)\big)\,dx = \|\vec{h}_-\|_1^2 \tag{1.16}$$

and

$$\|\vec{h}_+\|_1 = \|\vec{h}_-\|_1, \tag{1.17}$$

where $F'(u) = f(u)$ *with* $f(0) = 0$ *and*

$$\nu = \begin{cases} \frac{1}{2s+6} & \text{for } s \geqslant \frac{3}{2}, \\ \frac{2s-1}{2(2s+6)} & \text{for } s \leqslant \frac{3}{2}. \end{cases}$$

The rest of this paper is organized as follows. In next section, we obtain some estimates of (1.8) and give the proof of Theorem 1.2. In Section 3, we present the proof of Theorem 1.3 related to the nonlinear scattering for small amplitude solutions of (1.8).

2. Some preliminaries

In this section, we derive some estimates of (1.8) and give the proof of Theorem 1.2.

Lemma 2.1. *(See [13].) Let h be smooth and either convex or concave on* $[a, b]$ *with* $-\infty \leqslant a < b \leqslant +\infty$. *Then*

$$\left|\int_a^b e^{ih(\xi)}\right| \leqslant 4\left\{\min_{[a,b]} |h''|\right\}^{-1/2}$$

for $h'' \neq 0$ *in* $[a, b]$.

Lemma 2.2. *(See [24].) For any* n, $t > 1$ *and* $\varepsilon < 1$, *we have*

$$\sup_{\alpha\in R}\left|\int_{|\xi|\leqslant n} e^{ith(\xi,\alpha)}\,d\xi\right| \leqslant C_0 t^{-\frac{1}{2}} \max\left\{n^3, \varepsilon^{-\frac{7}{2}}\right\} + 2\varepsilon, \tag{2.1}$$

where $h(\xi, \alpha) = \frac{(1+\xi^2)^{\frac{1}{2}}\xi}{(1+\xi^2+\xi^4)^{\frac{1}{2}}} + \alpha\xi$, *and C is a constant.*

Lemma 2.3. *(See [24].) Let* $S(t)$ *be a* C_0 *group of unitary operators for the linearized equation*

$$\vec{u}_t - A\vec{u} = 0, \tag{2.2}$$

with $\vec{u}(0) = \vec{u}_0 = (u_0, v_0)$, *where*

$$A = \begin{pmatrix} 0 & \partial_x \\ \frac{(I-\partial_x^2)\partial_x}{I-\partial_x^2+\partial_x^4} & 0 \end{pmatrix},$$

then

$$\left|\vec{u}(t)\right|_\infty \leqslant C_0(1+t)^{-\nu}\left(\|\vec{u}_0\|_{Y_s} + |u_0|_1 + \left|\wedge^1 v_0\right|_1\right)$$

where $s > \frac{1}{2}$, $\vec{u}_0 \in Y_s$, $u_0 \in L^1$, $\wedge^1 v_0 \in L^1$, *and*

$$\nu = \begin{cases} \frac{1}{2s+6} & \textit{for } s \geqslant \frac{3}{2}, \\ \frac{2s-1}{2(2s+6)} & \textit{for } s \leqslant \frac{3}{2}, \end{cases}$$

and C_0 is a constant depending only on s.

Proof of Theorem 1.2. Let $\vec{w}(t) = S(t)\vec{u}_0$, then $\vec{w}(t)$ satisfies

$$\partial_t \vec{w} + \begin{pmatrix} 0 & -1 \\ \frac{(1+\xi^2)}{1+\xi^2+\xi^4} & 0 \end{pmatrix} \vec{w}_x = 0$$

with $\vec{w}(0) = \vec{u}_0$, and the solution $\vec{u}(t)$ of the nonlinear system of equations (1.8) can be written as

$$\vec{u}(t) = \vec{w}(t) + \partial_x \int_0^t S(t-\tau) \begin{pmatrix} 0 \\ -f(u) \end{pmatrix} d\tau.$$

Set

$$\vec{U}(x,t) = \int_{-\infty}^{x} \vec{u}(y,t)\,dy, \qquad \vec{U}_0(x) = \int_{-\infty}^{x} \vec{u}_0(y)\,dy, \qquad \vec{W}(x,t) = \int_{-\infty}^{x} \vec{w}(y,t)\,dy,$$

then

$$\vec{U}(x,t) = \vec{W}(x,t) + \vec{Z}(x,t), \tag{2.3}$$

where

$$\vec{Z}(x,t) = \int_0^t S(t-\tau) \begin{pmatrix} 0 \\ -f(u(\tau)) \end{pmatrix} d\tau.$$

First, it follows from the equation for $\vec{w}$ that

$$\vec{w}(x,t) = \vec{u}_0 - \partial_x \int_0^t \begin{pmatrix} 0 & -1 \\ \frac{(I-\partial_x^2)\partial_x}{I-\partial_x^2+\partial_x^4} & 0 \end{pmatrix} \vec{w}(\tau)\,d\tau.$$

Therefore

$$\vec{W}(x,t) = \vec{U}_0 - \int_0^t S(\tau) \left(\begin{pmatrix} 0 & -1 \\ \frac{(I-\partial_x^2)\partial_x}{I-\partial_x^2+\partial_x^4} & 0 \end{pmatrix} \vec{u}_0 \right) d\tau.$$

Using Lemma 2.3, we obtain

$$\begin{aligned}\left|\vec{W}(x,t)\right| &\leqslant |\vec{u}_0|_{L^1\times L^1} + C_0\int_0^t (1+\tau)^{-\nu}\,d\tau\left(\|\vec{u}_0\|_{Y_s} + |u_0|_1 + \left|\Lambda^1 v_0\right|\right)\\ &\leqslant C_0(1+t)^{1-\nu}\left(\|\vec{u}_0\|_{Y_s} + |u_0|_1 + \left|\Lambda^1 v_0\right|_1\right).\end{aligned}$$

Secondly, by Lemma 2.3 and the Cauchy Schwartz inequality, we get

$$\left|\vec{Z}(x,t)\right| \leqslant C_0\int_0^t \left(1+|t-\tau|^{-\nu}\right)\left|f(u)\right|_1\,d\tau, \tag{2.4}$$

and

$$\begin{aligned}\left|\vec{Z}(x,t)\right| &\leqslant C_0\int_0^t\left(\int_{-\infty}^{\infty}\left(1+\frac{\sqrt{1+\xi^2+\xi^4}}{\sqrt{1+\xi^2}}\right)\left|\widehat{f(u)}\right|d\xi\right)d\tau\\ &\leqslant C_0\int_0^t \left\|f\left(u(\cdot.\tau)\right)\right\|_1\,d\tau.\end{aligned} \tag{2.5}$$

Since $H^1\subset L^\infty$, $|f(s)| = O(|s|^p)$ and $|f'(s)| = O(|s|^{p-1})$ as $s\to 0$ for $p>1$, $|f(u)|_1 \leqslant C_0$ provided $p\geqslant 2$, where C_0 depends only on f and $\sup_{0\leqslant t\leqslant T}\|\vec{u}(t)\|_{Y_1}$. Thus, if $p\geqslant 2$,

$$\left|\vec{Z}(x,t)\right| \leqslant C_0\int_0^t \left(1+(t-\tau)\right)^{-\nu}d\tau \leqslant C_0\left(|t+1|^{1-\nu}\right).$$

If $1<p<2$, it is easy to show that $\|f(u)\|_{1,2/p}\leqslant C_0$. Since (2.4) and (2.5) hold for any $f\in L^1\cap H^1$, we can apply the interpolation theorem to the mapping $S(t-\tau)$ as in the estimate (2.4) and (2.5), and obtain

$$\left|\vec{Z}(x,t)\right| \leqslant C_0\int_0^t \left(1+(t-\tau)\right)^{-\nu(p-1)}\left\|f\left(u(\cdot,\tau)\right)\right\|_{1,2/p}\,d\tau \leqslant C_0|1+t|^{1-\nu(p-1)}.$$

Combining the estimates of $\vec{Z}(x,t)$ and $\vec{W}(x,t)$ yields the results of Theorem 1.2. □

In the rest of this section, we introduce the following lemma, which play an important role in the proof of Theorem 1.3.

Lemma 2.4. *Under the conditions of Lemma* 2.3*, if $\vec{u}_0\in Y_{k,1}\cap Y_{k+s_1,1}$ for some $s_1>\frac{1}{2}$ and $k\in R$, then $\vec{u}(t)\in Y_{k,\infty}$ and satisfies*

$$\left\|\vec{u}(t)\right\|_{k,\infty} \leqslant C\left(\|\vec{u}_0\|_{k+s_1,1} + \|\vec{u}_0\|_{k,1}\right)\left(1+|t|\right)^{-\nu_1}. \tag{2.6}$$

If $\vec{u}_0\in Y_{k+s_0+1/2,1}$ for some $s_0>\frac{1}{2}$ and $k+s_0>1$, $s_0>s_1$, then

$$\left\|\vec{u}(t)\right\|_{k,\infty} \leqslant C\left(1+|t|\right)^{-\nu_0}\|\vec{u}_0\|_{k+s_0+1/2,1}. \tag{2.7}$$

Moreover, $\vec{u}(t)$ satisfies

$$\big\|\vec{u}(t)\big\|_{k,p'} \leqslant C\big(1+|t|\big)^{-\nu_0(1-2/(p+1))}\|\vec{u}_0\|_{\tau(s_0,k),p}, \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $\tau(s_0,k)=\theta(s_0+k+1/2)+(1-\theta)k$, $\theta=\frac{1}{p}-\frac{1}{p'}$, C is a constant and

$$\nu_i = \begin{cases} \frac{1}{2s_i+6} & \text{for } s_i \geqslant \frac{3}{2}, \\ \frac{2s_i-1}{2(2s_i+6)} & \text{for } s_i \leqslant \frac{3}{2}, \end{cases}$$

where $i=0,1$.

Proof of Lemma 2.4. The solution $\vec{u}(t)=S(t)\vec{u}_0$ of (1.8) can be written as

$$\begin{aligned}\vec{u}(t) &= S(t)\vec{u}_0 \\ &= C\int_{-\infty}^{\infty} e^{ix\xi}\begin{pmatrix} \cos(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t) & i\frac{(1+\xi^2+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\sin(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t) \\ i\frac{\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}\sin(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t) & \cos(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t)\end{pmatrix}\widehat{\vec{u}_0}\,d\xi,\end{aligned} \tag{2.9}$$

where $\widehat{\vec{u}_0}$ is the Fourier transform of $\vec{u}_0$ and $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. Since

$$\Lambda^k\big|\vec{u}(t)\big| = \int_{-\infty}^{\infty} e^{ix\xi}\begin{pmatrix} \cos(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t) & i\frac{(1+\xi^2+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\sin(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t) \\ i\frac{\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}\sin(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t) & \cos(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}}t)\end{pmatrix}\widehat{\Lambda^k\vec{u}_0}\,d\xi, \tag{2.10}$$

it follows that

$$\begin{aligned}\big\|\vec{u}(t)\big\|_{k,\infty} &\leqslant C\sum\left|\int_{-\infty}^{\infty}\left(\widehat{\Lambda^k u_0}\pm\frac{(1+\xi^2+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\widehat{\Lambda^k v_0}\right)e^{it(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}})\pm x\xi/t}\,d\xi\right| \\ &\leqslant C\sum\left|\int_{|\xi|>n}\left(\widehat{\Lambda^k u_0}\pm\frac{(1+\xi^2+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\widehat{\Lambda^k v_0}\right)e^{it(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}})\pm x\xi/t}\,d\xi\right| \\ &\quad + C\sum\left|\int_{|\xi|<n}\left(\widehat{\Lambda^k u_0}\pm\frac{(1+\xi^2+\xi^4)^{\frac{1}{2}}}{\langle\xi\rangle}\widehat{\Lambda^k v_0}\right)e^{it(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}})\pm x\xi/t}\,d\xi\right| \\ &\leqslant C\|\vec{u}_0\|_{k+s_1}\left(\int_{|\xi|>n}\big(1+\xi^2\big)^{-s_1}\,d\xi\right)^{1/2} \\ &\quad + C\sum\int_{-\infty}^{+\infty}\big|\Lambda^k u_0(y)\pm\Lambda^k v_0(y)\big|\,dy\left|\int_{|\xi|\leqslant n} e^{it(\frac{\xi\langle\xi\rangle}{(1+\xi^2+\xi^4)^{\frac{1}{2}}})\pm x\xi/t}\,d\xi\right|,\end{aligned}$$

where the summation ($\sum$) is over all two sign combinations.

It follows from Lemma 2.2 that

$$\begin{aligned}\big\|\vec{u}(t)\big\|_{k,\infty} &\leqslant C\|\vec{u}_0\|_{Y_{k+s_1}} n^{-(s_1-1/2)} \\ &\quad + C\big(t^{-1/2}\max\big\{n^{7/2}, \varepsilon^{-1}\big\} + \varepsilon + t^{-1/2}\varepsilon^{-1/2}\big)\|\vec{u}_0\|_{k,1}.\end{aligned}$$

Choosing $\varepsilon = t^{-\alpha}$, $n = t^{\alpha}$, and letting $\alpha(s_1 - \frac{1}{2}) = \frac{1}{2} - \frac{7}{2}\alpha$, we get

$$\begin{aligned}\big\|\vec{u}(t)\big\|_{k,\infty} &\leqslant C\big(\|\vec{u}_0\|_{k+s} + \|\vec{u}_0\|_{k,1}\big)\big(t^{-\alpha(s_1-1/2)} \\ &\quad + |t|^{-(1/2-7/2\alpha)} + |t|^{-\alpha} + |t|^{-(1/2-1/2\alpha)}\big) \\ &\leqslant C\big(\|\vec{u}_0\|_{k+1} + \|\vec{u}_0\|_{k,s_1}\big)|t|^{-\nu_1},\end{aligned} \tag{2.11}$$

where

$$\nu_1 = \begin{cases} \frac{1}{2s_1+6} & \text{for } s_1 \geqslant \frac{3}{2}, \\ \frac{2s_1-1}{2(2s_1+6)} & \text{for } s_1 \leqslant \frac{3}{2}. \end{cases}$$

On the other hand, for $|t| \leqslant 1$, we have

$$\big\|\vec{u}(t)\big\|_{k,\infty} \leqslant C\|\vec{u}_0\|_{k+s_1}. \tag{2.12}$$

Combining (2.11) for $|t| \geqslant 1$ with (2.12) for $t \leqslant 1$, the proof of the estimate (2.6) is complete.

Since the linear operator $S(t)$ is unitary in Y_k for any $k \in R$, it follows that

$$\big\|\vec{u}(t)\big\|_k = \big\|S(t)\vec{u}_0\big\|_k = \|\vec{u}_0\|_k. \tag{2.13}$$

By Sobolev's embedding theorem $L^1_{k+s_0+\frac{1}{2}+i} \to L^2_{k+s_2+i}$ for $s_0 > s_2, s_2 + k \geqslant 1$ and $i = 0, 1$ to (2.6), we have

$$\big\|\vec{u}(t)\big\|_{k,\infty} \leqslant C\big(1 + |t|\big)^{-\nu_0}\|\vec{u}_0\|_{k+s_0+\frac{1}{2},1}. \tag{2.14}$$

Applying the interpolation theorem for the operator $S(t)$ to (2.7), we get (2.8). The proof of Lemma 2.4 is complete. □

3. Proof of Theorem 1.3

In this section, we consider the scattering of the small solutions for (1.8), and give the proof of Theorem 1.3.

Proof of Theorem 1.3. We shall prove Theorem 1.3 by four steps below.

Step 1. Consider the existence of solution $\vec{u}$ for any $-\infty \leqslant s \leqslant +\infty$. Let $V(\delta_1) = \{\vec{u} \in V \mid \|\vec{u}\|_V \leqslant \delta_1\}$ be a complete metric subspace of V, and T be a map defined as follows

$$T\vec{u}(t) = S(t)\vec{h}_- + \int_s^t S(t-\tau)F\big(\vec{u}(\tau)\big)\,d\tau \quad \text{for all } -\infty \leqslant s \leqslant +\infty. \tag{3.1}$$

It follows from Lemma 2.4 that

$$
\begin{aligned}
\left\|T\vec{u}(t)\right\|_{1,p+1} &\leqslant \left\|S(t)\vec{h}_-\right\|_{1,p+1} + C\left|\int_s^t \left(1+|t-\tau|\right)^{-\nu}\left\|\frac{\partial_x}{I-\partial_x^2+\partial_x^4}\vec{f}\left(\vec{u}(\tau)\right)\right\|_{\tau(s,k),q} d\tau\right| \\
&= \left\|S(t)\vec{h}_-\right\|_{1,p+1} + C\left|\int_s^t \left(1+|t-\tau|\right)^{-\nu}\left\|\frac{\partial_x}{I-\partial_x^2+\partial_x^4} f\left(u(\tau)\right)\right\|_{\tau(s,k)+1,q} d\tau\right| \\
&\leqslant \left\|S(t)\vec{h}_-\right\|_{1,p+1} + C\left|\int_s^t \left(1+|t-\tau|\right)^{-\nu}\left\|\vec{u}(\tau)\right\|_{1,p+1}^p d\tau\right|,
\end{aligned}
\tag{3.2}
$$

where $\frac{1}{q}+\frac{1}{p+1}=1$, $\tau(s,k)=\theta(k+s+1/2)+(1-\theta)k$, $\theta=\frac{1}{q}-\frac{1}{p+1}$.

Therefore, we get

$$
\left(1+|t|\right)^{\nu}\left\|T\vec{u}(t)\right\|_{1,p+1} \leqslant \left(1+|t|\right)^{\nu}\left\|S(t)\vec{h}_-\right\|_{1,p+1} + CI\|\vec{u}\|_V^p,
\tag{3.3}
$$

where

$$
I = \sup_{t,s\in R}\left(1+|t|\right)^{\nu}\left|\int_s^t \left(1+|t-\tau|^{\nu}\right)\left(1+|\tau|\right)^{-\nu p} d\tau\right|.
$$

Since $\nu(1-\frac{2}{p+1})p>1$, the integral is $O(1+|t|)^{-\nu}$ as $|t|\to\infty$, and I is bounded uniformly.

Furthermore, we have

$$
\begin{aligned}
\left\|T\vec{u}(t)\right\|_1 &\leqslant \left\|S(t)\vec{h}_-\right\|_1 + \left|\int_s^t \left\|S(t-\tau)\frac{\partial_x}{I-\partial_x^2+\partial_x^4}\vec{f}\left(\vec{u}(\tau)\right)\right\|_1 d\tau\right| \\
&\leqslant \left\|S(t)\vec{h}_-\right\|_1 + C\left|\int_s^t \left\|\frac{\partial_x}{I-\partial_x^2+\partial_x^4} f\left(u(\tau)\right)\right\|_1 d\tau\right| \\
&\leqslant \left\|S(t)\vec{h}_-\right\|_1 + C\left|\int_s^t \left\|\vec{u}(\tau)\right\|_{1,p+1}^p\right| \\
&\leqslant \left\|S(t)\vec{h}_-\right\|_1 + C\int_{-\infty}^{+\infty}\left(1+|\tau|\right)^{-\nu} d\tau\,\|\vec{u}\|_V^p \\
&\leqslant \left\|S(t)\vec{h}_-\right\|_1 + C\|\vec{u}\|_V^p.
\end{aligned}
\tag{3.4}
$$

Combining (3.3) with (3.4) yields

$$
\|T\vec{u}\|_V \leqslant \left\|S(\cdot)\vec{h}_-\right\|_V + C(I+1)\|\vec{u}\|_V^p.
\tag{3.5}
$$

On the other hand, if $\vec{u},\vec{v}\in V(\delta_1)$, $\vec{u}=(u,u_1)$, $\vec{v}=(v,v_1)$, then a similar estimate to (3.2) is obtained for the difference $(T\vec{u}-T\vec{v})$ as follows

$$\begin{aligned}\big\|(T\vec{u}-T\vec{v})\big\|_{1,p+1} &\leqslant \left|\int_s^t \big(1+|t-\tau|\big)^{-\nu}\left\|\frac{\partial_x}{I-\partial_x^2+\partial_x^4}\big(f\big(u(\tau)\big)-f\big(v(\tau)\big)\big)\right\|_{\tau(s,k)+1,q} d\tau\right| \\ &\leqslant C\left|\int_s^t \big(1+|t-\tau|\big)^{-\nu}\big(\|\vec{u}\|_{1,p+1}^{p-1}+\|\vec{v}\|_{1,p+1}^{p-1}\big)\|\vec{u}-\vec{v}\|_{1,p+1}\,d\tau\right|.\end{aligned}$$

Hence

$$\big(1+|t|\big)^{\nu}\big\|(T\vec{u}-T\vec{v})(T)\big\|_{1,p+1} \leqslant CI\big(\|\vec{u}\|_V^{p-1}+\|\vec{v}\|_V^{p-1}\big)\|\vec{u}-\vec{v}\|_V. \tag{3.6}$$

We have also a similar estimate in Y_1

$$\begin{aligned}\big\|(T\vec{u}-T\vec{v})(t)\big\|_1 &\leqslant C\left|\int_s^T \big|f_x(u)-f_x(v)\big|_2\,d\tau\right| \\ &\leqslant C\int_{-\infty}^{\infty}\Big|\big(|u|^{p-2}+|v|^{p-2}\big)|u-v|_{\frac{2(p+1)}{p-1}}|u_x|_{p+1}+|v|_{2(p+1)}^{p-1}|u_x-v_x|_{p+1}\Big|\,d\tau \\ &\leqslant C\int_{-\infty}^{\infty}\Big[|u-v|_\infty|u_x|_{p+1}\big(|u|^{p-2}_{\frac{2(p+1)(p-2)}{p-1}}+|v|^{p-2}_{\frac{2(p+1)(p-2)}{p-1}}\big)+|v|_{2(p+1)}^{p-1}|u_x-v_x|_{p+1}\Big]\,d\tau \\ &\leqslant C\int_{-\infty}^{\infty}\big(\big\|u(\tau)_{1,p+1}^{p-1}\big\|+\big\|v(\tau)_{1,p+1}^{p-1}\big\|\big)\big\|(u-v)(\tau)\big\|_{1,p+1}\,d\tau \\ &\leqslant C\big(\|\vec{u}\|_V^{p-1}+\|\vec{v}\|_V^{p-1}\big)\|\vec{u}-\vec{v}\|_V.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\|T\vec{u}-T\vec{v}\|_V \leqslant C(I+1)\big(\|\vec{u}\|_V^{p-1}+\|\vec{v}\|_V^{p-1}\big)\|\vec{u}-\vec{v}\|_V. \tag{3.8}$$

Therefore, if we choose $\delta \leqslant \frac{\delta_1}{2}$ and δ_1 so small that $2C(I+1)\delta_1^{p-1} \leqslant 1/2$, then it follows from (3.5) and (3.8) that $T: V(\delta_1) \to V(\delta_1)$, and is a strict contraction map. So it has a fixed point $\vec{u} \in V(\delta_1)$ which is a solution of (1.9) in $V(\delta_1)$. It is easy to prove uniqueness, we omit it.

Step 2. We will derive (1.16), and rewrite (1.9) as

$$\vec{u}_s(t) = S(t)\vec{u}_{0,s} + \int_0^t S(t-\tau)\vec{f}\big(\vec{u}_s(\tau)\big)\,d\tau, \tag{3.9}$$

where $\vec{u}_{0,s} = \vec{h}_- - \int_0^s S(-\tau)\vec{f}(\vec{u}_s(\tau))\,d\tau$.

We find

$$\|\vec{u}_{0,s}\| \leqslant \|\vec{h}_-\|_1 + \int_0^s \big|f_x\big(u(\tau)\big)\big|_2\,d\tau \leqslant \|\vec{h}_-\|_1 + C\|\vec{u}\|_V^p \leqslant C\delta_1 + C\delta_1^p, \tag{3.10}$$

where the constant C is independent of s.

Thus, if δ_1 is chosen to be small sufficiently, by the uniqueness of solution of (1.8), we get

$$\begin{aligned}\|\vec{u}_s(t)\|_1^2 - 2\int_{-\infty}^{\infty} F\big(u_s(t)\big)\,dx &= \|\vec{u}_s(s)\|_1^2 - 2\int_{-\infty}^{\infty} F\big(u_s(s)\big)\,dx \\ &= \|S(s)\vec{h}_-\|_1^2 - 2\int_{-\infty}^{\infty} F\big(S(s)\vec{h}_-\big)\,dx. \qquad (3.11)\end{aligned}$$

It follows from (2.8) in Lemma 2.4 that

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} F\big(S(s)\vec{h}_-\big)\,dx\right| &\leqslant C\int_{-\infty}^{+\infty} \big|S(s)\vec{h}_-\big|^{p+1} \leqslant C\|S(s)\vec{h}_-\|_{1,p+1}^{p+1} \\ &\leqslant C\big[\big(1+|t|\big)^{-\nu}\big]^{p+1}\|\vec{h}_-\|_{1,q}^{p+1} \to 0, \qquad (3.12)\end{aligned}$$

as $s \to \pm\infty$. Hence

$$\lim_{s\to-\infty}\left(\|\vec{u}_s(t)\|_1^2 - 2\int_{-\infty}^{+\infty} F\big(\vec{u}_s(t)\big)\,dx\right) = \|\vec{h}_-\|_1^2. \qquad (3.13)$$

On the other hand, let $\vec{u}$ be the solution of (1.9). Since $\|\vec{u}_s\|_V \leqslant \delta_1$ for $-\infty \leqslant s \leqslant +\infty$, and δ_1 is independent of s, we have

$$\begin{aligned}\|\vec{u}_s(t) - \vec{u}(t)\|_1 &\leqslant \left|\int_s^t \big|f_x\big(u_s(\tau)\big) - f_x\big(u(\tau)\big)\big|_2\,d\tau\right| + \int_{-\infty}^{s} \big|f_x\big(u(\tau)\big)\big|_2\,d\tau \\ &\leqslant C\big(\|\vec{u}_s\|_V^{p-1} + \|\vec{u}\|_V^{p-1}\big)\|\vec{u}_s - \vec{u}\|_V + C\int_{-\infty}^{s}\big(1+|\tau|\big)^{-\nu p}\|\vec{u}\|_V^p \\ &\leqslant C\delta_1^{p-1}\|\vec{u}_s - \vec{u}\|_V + C\delta_1^p\varepsilon(s), \qquad (3.14)\end{aligned}$$

where $\varepsilon(s) = \int_{-\infty}^{s}(1+|\tau|)^{-\nu}\,d\tau \to 0$ as $s \to -\infty$, and

$$\big(1+|t|\big)^{\nu}\|\vec{u}_s(t) - \vec{u}(t)\|_{1,p+1} \leqslant CI\big(\|\vec{u}_s\|_V^{p-1} + \|\vec{u}\|_V^{p-1}\big)\|\vec{u}_s - \vec{u}\|_V + C\eta(s)\|\vec{u}\|_V^p, \qquad (3.15)$$

where

$$\eta(s) = \sup_{t\in R}\big(1+|t|\big)^{\nu}\int_{-\infty}^{s}\big(1+|t-\tau|\big)^{-\nu}\big(1+|\tau|\big)^{-\nu}\,d\tau, \qquad (3.16)$$

where $\eta(s) \to 0$ as $s \to -\infty$.

Thus

$$\|\vec{u}_s - \vec{u}\|_V \leqslant 2C\delta_1^{p-1}\|\vec{u}_s - \vec{u}\|_V + C\delta_1^p\big(\varepsilon(s) + \eta(s)\big). \qquad (3.17)$$

Choosing δ_1 so small that $2C\delta_1^{p-1} \leqslant 1/2$, we obtain $\|\vec{u}_s - \vec{u}\|_V \to 0$ as $s \to -\infty$. Also, we have

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \big(F\big(u_s(t)\big) - F\big(u(t)\big)\big)\,dx\right| &\leqslant C\int_{-\infty}^{+\infty} \big(\big|u_s(t)\big|^p + \big|u(t)\big|^p\big)\big|u_s(t) - u(t)\big|\,dx \\ &\leqslant C\big(\big|u_s(t)\big|_{p+1}^p + \big|u(t)\big|_{p+1}^p\big)\big|u_s(t) - u(t)\big|_{p+1} \\ &\leqslant C\big(\|\vec{u}_s\|_V^p + \|\vec{u}\|_V^p\big)\|\vec{u}_s - \vec{u}\|_V \\ &\leqslant 2C\delta_1^p\|\vec{u}_s - \vec{u}\|_V \to 0, \end{aligned} \tag{3.18}$$

as $t \to -\infty$.

Taking the limit in (3.13), the proof of (1.16) is complete.

Step 3. We will show the asymptotic behavior (1.14). Since $\vec{u} \in V(\delta_1)$, we have

$$\begin{aligned}\big\|\vec{u}(t) - S(t)\vec{h}_-\big\|_1 &\leqslant \int_{-\infty}^{t} \big\|\vec{f}\big(u(\tau)\big)\big\|_1\,d\tau \leqslant C\int_{-\infty}^{t} \big\|u(\tau)\big\|_{1,p+1}^p\,d\tau \\ &\leqslant C\int_{-\infty}^{t} \big(1+|\tau|\big)^{-\nu(1-\frac{2}{p+1})p}\,d\tau\,\|\vec{u}\|_V^p \\ &\leqslant C\delta_1^p\int_{-\infty}^{t} \big(1+|\tau|\big)^{-\nu(1-\frac{2}{p+1})p}\,d\tau \to 0, \end{aligned} \tag{3.19}$$

as $t \to -\infty$, since $\nu(1 - \frac{2}{p+1})p > 1$.

Step 4. In order to prove the existence of $\vec{h}_+ \in Y_1$, (1.15) and (1.17), we define

$$\vec{h}_+ = \vec{h}_- + \int_{-\infty}^{+\infty} S(-\tau)\vec{f}\big(\vec{u}(\tau)\big)\,d\tau. \tag{3.20}$$

Since the solution $\vec{u}$ of (1.9) is in $V(\delta_1)$ and $\vec{h}_- \in Y_1$, we have

$$\begin{aligned}\|\vec{h}_+\| &\leqslant \|\vec{h}\|_1 + C\int_{-\infty}^{+\infty} \big(1+|\tau|\big)^{-\nu}\left(1 - \frac{2}{p+1}\right)^p d\tau\,\|\vec{u}\|_V^p \\ &\leqslant \|\vec{h}_-\|_1 + 2C\delta_1^p, \end{aligned} \tag{3.21}$$

which implies $\vec{h}_+ \in Y_1$.

To prove (1.17), first of all, we show $\|S(\cdot)\vec{h}_+\|_V < \frac{\delta_1}{2}$. In fact, it follows from (3.20) that

$$\big(1+|t|\big)^{\nu(1-\frac{2}{p+1})}\big\|S(t)\vec{h}_+\big\|_{1,p+1} \leqslant \big\|S(\cdot)\vec{h}_-\big\|_V + CI\|\vec{u}\|_V^p, \tag{3.22}$$

where I is defined as above.

Therefore, by (3.21) and (3.22), we obtain

$$\left\| S(\cdot)\vec{h}_+ \right\|_V \leqslant \left\| S(\cdot)\vec{h}_- \right\|_V + C(I+1)\delta_1^p. \tag{3.23}$$

If we choose $\delta < \frac{\delta_1}{4}$ and $C(I+1)\delta_1^{p-1} < \frac{1}{4}$, then

$$\left\| S(\cdot)\vec{h}_+ \right\|_V < \frac{\delta_1}{2}. \tag{3.24}$$

Consider the integral equation

$$\vec{u}_s(t) = S(t)\vec{h}_+ + \int_s^t S(t-\tau)\vec{f}\big(\vec{u}_s(\tau)\big)\,d\tau \tag{3.25}$$

with $-\infty \leqslant s \leqslant +\infty$.

Similar to the proof of existence of $\vec{u}_s$ for (1.9), we can also show that (3.25) has a unique solution $\vec{u}_s \in V(\delta_1)$, and rewrite it as

$$\vec{u}_s(t) = S(t)\vec{u}_{0,s} + \int_0^t S(t-\tau)\vec{f}(\vec{u}_s)\,d\tau, \tag{3.26}$$

where $\vec{u}_{0,s} = \vec{h}_+ - \int_0^s S(-\tau)\vec{f}(\vec{u}(\tau))\,d\tau$.

Similar to the proof of (1.16), one can obtain

$$\left\| u(t) \right\|_1^2 + \left\| v(t) \right\|_2^2 - 2\int_{-\infty}^{\infty} F\big(u(t)\big)\,dx = \|\vec{h}_+\|_1^2, \tag{3.27}$$

where $\vec{u}$ is the solution of (3.25). This implies (1.17).

The proof of (1.15) is similar to that of (1.14), and we omit it. This completes the proof of Theorem 1.3. □

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